Graph Theory Definitions

Basic Definitions

Vertex A point, node, or vertex is a non-specific location. It can be moved around at will and usually is labeled with some name for convenience. A set of vertices can be re-labeled at will. $v \in V$

Edge An edge is a two-way connection between two vertices. Edges are denoted as $e(v, u)$ and the set of edges is usually denoted as $e \in E$.

Arc An arc is a directional connection between two vertices. Arcs are denoted as $a(v, u)$. The existence of $a(v, u)$ does not imply that $a(u, v)$ exists.

Graph A graph $G(V, E)$ is a non-empty set of vertices V and a set of edges E. The vertices can be re-labeled at will.

Digraph A Digraph $G(V, A)$ is a non-empty set of vertices V and a set of arcs A. The vertices can be re-labeled at will.

Graph to Digraph For this class, a graph includes only edges, not arcs. If both are present, replace each edge $e(u, v)$ with two arcs $e(u, v)$ and $e(v, u)$ to produce a Digraph.

Other Definitions

Adjacency List A graph(digraph) may be completely represented by a list of the verticies an their adjacent vertices (neighbors).

Adjacency Matrix A graph(digraph) can be completely represented by a square matrix with the vertices listed for both dimensions. If an edge(arc) exists, that entry would be a 1, otherwise it would be a zero.

Adjacent Two vertices v and u are adjacent if, and only if, $e(v, u)$ exists. In the case of arcs, two vertices v and u are adjancent if, and only if, either $a(v, u)$ or (u, v) exists.

Bipartite Graph A graph whose vertices can be separated into two disjoint sets such that each edge has an endpoint in each set. Any bipartite graph can be colored with two colors.

Chain A chain is a sequence of arcs that connects two vertices . A chain between u and v does not imply a chain between v and u. (See also Path. Some sources flip the meaning of path and chain.)

Chromatic Number The chromatic number of a graph is the smallest number of colors required to color a graph.

Circuit A chain in a digraph that ends at the starting vertex. Strictly speaking, this violates the meaning of chain, but we will allow that in this case.

Coloring The vertices of a graph may be colored in such a way that no edge connects two vertices of the same color. The minimum number of colors required to color a graph is the graph's chromatic number.

Complement of a Graph The complement of a graph contains the same set of vertices but replaces the set of edges with all of the missing edges. In otherwords if G' is the complement of G then $e \in G(V, E) \leftrightarrow e \notin G'(V, E')$. Some authors use $\overline{G}$ to represent the complement of G.

Complete Graph A graph with all possible edges included. Such a graph is completely defined by the number of vertices, n, and is referred to as K_n from the German *komplete(sp?)*

Connected Diraph A digraph is said to be connected if there exists a chain between any two arbitrary vertices u and v in either one direction or the other. If for any two vertices there exists a chain in either direction, the digraph is said to be strongly connected.

Connected Graph A graph is said to be connected if there exists a path between any two arbitrary vertices u and v.

Cycle A path(chain) of at least three edges(arcs) that includes a vertex more than once. A graph(digraph) with no cycles is said to be acyclic. We need not make a distinction between circuit and cycle.

Degree of a Graph The degree of a graph is the sum of the degrees of all the vertices and must be even. (Why? This is left to the student as an exercise.)

Degree of a Vertex The degree of a vertex is the number of edges that is incident on that vertex and is denoted as $\deg(v)$. A vertex in a digraph has both “in” and “out” degree.

Distance in a Digraph The distance between two vertices v and u in a digraph is the minimun number of arcs that must be traversed over a simple chain to move from a vertex v to a vertex u. Obviously, the distance $v \rightsquigarrow u$ might not be the same as the distance $u \rightsquigarrow v$. The longest distance in a digraph is the diameter of the digraph.

Distance in a Graph The distance between two vertices v and u in a graph is the minimun number of edges that must be traversed over a simple path to move from a vertex v to a vertex u. The longest distance in a graph is the diameter of the graph.

Eulerian Chain or Path For our purposes, we will not make a distinction between these two. An Eulerian chain(path) touches each edge in a graph (or digraph) exactly once.

Euler(ian) Circuit or Cycle For our purposes, we will not make a distinction between these two. An Eulerian cycle touches each edge in a graph (or digraph) exactly once and returns to the starting vertex. If there is one Euler Cycle in a graph there many be many.

Hamiltonian Chain or Path A Hamiltonian chain or path touches each vertex in a graph(digraph) exactly once.

Hamiltonian Circuit or Cycle A Hamiltonian circuit or cycle touches each vertex in a graph(digraph) exactly once and returns to the starting vertex.

Loop A loop is any edge that begins and ends at the same vertex. Most of the graphs we will be interested in will not have loops.

Missing Edge, Non-edge, or Anti-edge If and edge e has endpoints $u, v \in V$ and $e \notin E$, then e is a missing edge or anti-edge.

Multiple Edge An edge is said to be multiple if there are more than one edge in the set E with the same two endpoints.

Order of a Graph The order of a graph G is $|V|$.

Path A path is a sequence of alternating vertices and edges that connects two vertices without repeating any vertex. (This is a simple path.)

Simple Graph A graph is said to be simple if there are no loops or edges that are multiples. Unless otherwise **explicitly stated**, in this class we are interested in simple graphs.

Size of a Graph The size of a graph G is $|E|$.

Strongly Connected Digraph A digraph is said to be strongly connected if, and only if, for any two arbitrary vertices u, v there exist chains $u \rightsquigarrow v$ and $v \rightsquigarrow u$.

Subgraph A graph $H(V_H, E_H)$ is a subgraph of $G(V_G, E_G)$ iff: $\{v \in V_H \rightarrow v \in V_G\} \wedge \{e \in E_H \rightarrow e \in E_G\}$. In other words, $V_H \subseteq V_G$ and $E_H \subseteq E_G$.

Tree An acyclic connected graph (digraph) or a connected graph (digraph) with the fewest possible number of edges. If any edge is removed from a tree the result is no longer a connected graph. Any vertex of a tree may be designated the "root" of a tree. Picking a new root does not change the graph.

Walk A walk is an alternating sequence of vertices and edges (arcs) that connects two vertices. A path is a walk but a walk might not be a path.

Weakly Connected Digraph A digraph is said to be weakly connected if, and only if, for any two arbitrary vertices u, v there exist chains $u \rightsquigarrow v \vee v \rightsquigarrow u$. If for all pairs u, v both chains exist, the digraph is strongly connected.