Graph Theory Definitions

Basic Definitions

Vertex A point, node, or vertex is a non-specific location. It can be moved around at will and usually is labeled with some name for convenience. A set of vertices can be re-labeled at will, \( v \in V \).

Edge An edge is a two-way connection between two vertices. Edges are denoted as \( e(v, u) \) and the set of edges is usually denoted as \( e \in E \).

Arc An arc is a directional connection between two vertices. Arcs are denoted as \( a(v, u) \). The existence of \( a(v, u) \) does not imply that \( a(u, v) \) exists.

Graph A graph \( G(V, E) \) is a non-empty set of vertices \( V \) and a set of edges \( E \). The vertices can be re-labeled at will.

Digraph A Digraph \( G(V, A) \) is a non-empty set of vertices \( V \) and a set of arcs \( A \). The vertices can be re-labeled at will.

Graph to Digraph For this class, a graph includes only edges, not arcs. If both are present, replace each edge \( e(u, v) \) with two arcs \( e(u, v) \) and \( e(v, u) \) to produce a Digraph.

Other Definitions

Adjacency List A graph(digraph) may be completely represented by a list of the vertices and their adjacent vertices (neighbors).

Adjacency Matrix A graph(digraph) can be completely represented by a square matrix with the vertices listed for both dimensions. If an edge(arc) exists, that entry would be a 1, otherwise it would be a zero.

Adjacent Two vertices \( v \) and \( u \) are adjacent if, and only if, \( e(v, u) \) exists. In the case of arcs, two vertices \( v \) and \( u \) are adjacent if, and only if, either \( a(v, u) \) or \( (u, v) \) exists.

Bipartite Graph A graph whose vertices can be separated into two disjoint sets such that each edge has an endpoint in each set. Any bipartite graph can be colored with two colors.

Chain A chain is a sequence of arcs that connects two vertices. A chain between \( u \) and \( v \) does not imply a chain between \( v \) and \( u \). (See also Path. Some sources flip the meaning of path and chain.)

Chromatic Number The chromatic number of a graph is the smallest number of colors required to color a graph.

Circuit A chain in a digraph that ends at the starting vertex. Strictly speaking, this violates the meaning of chain, but we will allow that in this case.

Coloring The vertices of a graph may be colored in such a way that no edge connects two vertices of the same color. The minimum number of colors required to color a graph is the graph’s chromatic number.

Complement of a Graph The complement of a graph contains the same set of vertices but replaces the set of edges with all of the missing edges. In other words, if \( G' \) is the complement of \( G \) then \( e \in G(V, E) \iff e \notin G'(V, E') \). Some authors use \( \overline{G} \) to represent the complement of \( G \). Also if \( G' \) is the complement of \( G \) then \( G(V, E) \rightarrow G'(V, \overline{E}) \).
**Complete Graph** A graph with all possible edges included. Such a graph is completely defined by the number of vertices, \( n \), and is referred to as \( K_n \) from the German *komplete*(sp?)

**Connected Digraph** A digraph is said to be connected if there exists a chain between any two arbitrary vertices \( u \) and \( v \) in either one direction or the other. If for any two vertices there exists a chain in either direction, the digraph is said to be strongly connected.

**Connected Graph** A graph is said to be connected if there exists a path between any two arbitrary vertices \( u \) and \( v \).

**Cycle** A path(chain) of at least three edges(arcs) that includes a vertex more than once. A graph(digraph) with no cycles is said to be acyclic. We need not make a distinction between circuit and cycle.

**Degree of a Graph** The degree of a graph is the sum of the degrees of all the vertices and must be even. (Why? This is left to the student as an exercise.)

**Degree of a Vertex** The degree of a vertex is the number of edges that is incident on that vertex and is denoted as \( \deg(v) \). A vertex in a digraph has both “in” and “out” degree.

**Distance in a Digraph** The distance between two vertices \( v \) and \( u \) in a digraph is the minimum number of arcs that must be traversed over a simple chain to move from a vertex \( v \) to a vertex \( u \). Obviously, the distance \( v \rightarrow u \) might not be the same as the distance \( u \rightarrow v \). The longest distance in a digraph is the diameter of the digraph.

**Distance in a Graph** The distance between two vertices \( v \) and \( u \) in a graph is the minimum number of edges that must be traversed over a simple path to move from a vertex \( v \) to a vertex \( u \). The longest distance in a graph is the diameter of the graph.

**Empty Graph (Digraph)** The special case of the graph (digraph) with no vertices, that is \( E, V \) are empty. In this class, it is considered a special case and not really a graph. This is analogous to the empty set \( \{\} \).

**Eulerian Chain or Path** For our purposes, we will not make a distinction between these two. An Eulerian chain(path) touches each edge in a graph (or digraph) exactly once.

**Euler(ian) Circuit or Cycle** For our purposes, we will not make a distinction between these two. An Eulerian cycle touches each edge in a graph (or digraph) exactly once and returns to the starting vertex. If there is one Euler Cycle in a graph there many be many.

**Hamiltonian Chain or Path** A Hamiltonian chain or path touches each vertex in a graph(digraph) exactly once.

**Hamiltonian Circuit or Cycle** A Hamiltonian circuit or cycle touches each vertex in a graph(digraph) exactly once and returns to the starting vertex.

**Loop** A loop is any edge that begins and ends at the same vertex. Most of the graphs we will be interested in will not have loops.

**Missing Edge, Non-edge, or Anti-edge** If and edge \( e \) has endpoints \( u, v \in V \) and \( e \notin E \), then \( e \) is a missing edge or anti-edge.

**Multiple Edge** An edge is said to be multiple if there are more than one edge in the set \( E \) with the same two endpoints.
Order of a Graph  The order of a graph $G$ is $|V|$.

Path  A path is a sequence of alternating vertices and edges that connects two vertices without repeating any vertex. (This is a simple path.)

Simple Graph  A graph is said to be simple if there are no loops or edges that are multiples. Unless otherwise explicitly stated, in this class we are interested in simple graphs.

Size of a Graph  The size of a graph $G$ is $|E|$.

Strongly Connected Digraph  A digraph is said to be strongly connected if, and only if, for any two arbitrary vertices $u, v$ there exist chains $u \leadsto v$ and $v \leadsto u$.

Subgraph  A graph $H(V_H, E_H)$ is a subgraph of $G(V_G, E_G)$ iff: $\{v \in V_H \rightarrow v \in V_G\} \land \{e \in E_H \rightarrow e \in E_G\}$. In other words, $V_H \subseteq V_G$ and $E_H \subseteq E_G$.

Tree  An acyclic connected graph (digraph) or a connected graph (digraph) with the fewest possible number of edges. If any edge is removed from a tree the result is no longer a connected graph. Any vertex of a tree may be designated the “root” of a tree. Picking a new root does not change the graph.

Spanning Tree  A tree constructed over a connected graph such that every node and edge of the tree is in the original graph. If the edges (arcs) are weighted, it is possible to construct a tree that spans the graph with the minimum total weight. This tree is a Minimum Spanning Tree or MST. If a spanning tree, or MST, exists there must be multiple spanning trees (each with a different root, for instance.)

Walk  A walk is an alternating sequence of vertices and edges (arcs) that connects two vertices. A path is a walk but a walk might not be a path.

Weakly Connected Digraph  A digraph is said to be weakly connected if, and only if, for any two arbitrary vertices $u, v$ there exist chains $u \leadsto v \lor v \leadsto u$. If for all pairs $u, v$ both chains exist, the digraph is strongly connected.